Application of Moment Expansion method to Options Square Root Model

Yun Zhou

Advisor: Dr. Steven Heston

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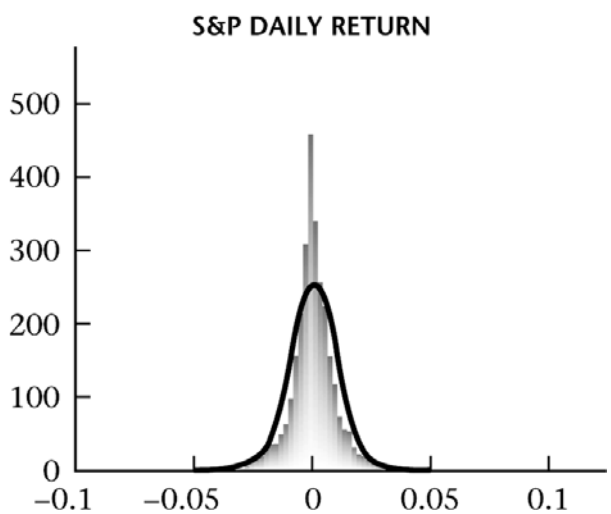
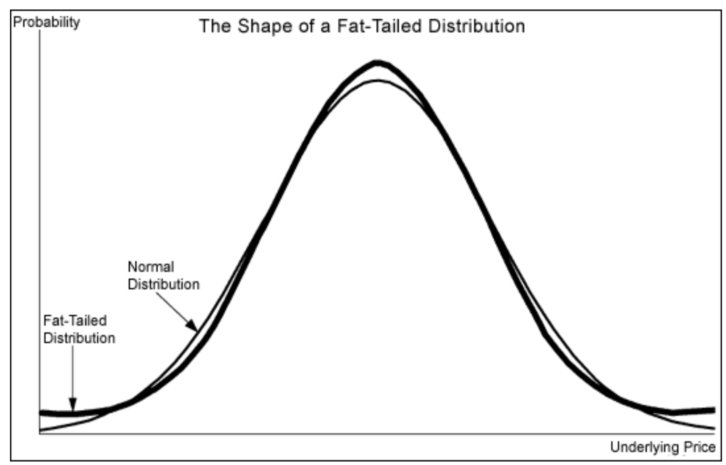
Background

- **Black–Scholes model**
  - Equity price at time t follows a geometric Brownian motion
  - Explain stock option prices
  - Easy implentation
  - 1997 Nobel Prize in Economics

Background

- Problem arise
  - ➢ Model Assumption: daily return is normal distribution, constant volatility
  - ➢ Reality: daily return are asymmetric with fat tails, volatility is not constant





Background

- Heston (1993) Square Root Model

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^s$$

$$dv_t = k(\theta - v_t)dt + \xi \sqrt{v_t} dW_t^v$$

$$dW_t^s dW_t^v = \rho dt$$

vt is the instantaneous variance
μ is the average rate of return of the asset.
θ is the long vol, or long run average price volatility
κ is the rate at which v*t* reverts to θ.
ξ is the vol of vol, or volatility of the volatility, i.e, the variance of v*t*.

Background

- European Call Option payoff

$$C(T,S(T)) = (S(T)-K)^{+}$$

T : expiration time

K: Strike price

- Boundary conditions

1) $c(t,0)=0$

2) $\lim_{s(t)\to\infty}[c(t,S(t))-(S(t)-e^{-r(T-t)}K)]=0$

for all $t\in[0,T]$

Approach

- Fourier Transform Based Solution: Heston(1993)

- New Approach here : Moment Expansion Solution

  - Get high moments by Moment generating function
  - Comparison with Fourier Transform Based solution to determine the Maximum order moment
  - Get option prices from truncated moment expansion

Approach

$$dx = (\mu - \tfrac{1}{2}v)dt + \sqrt{v}\, dW_t^s$$

$$x = \ln S(t)$$

- Moment Expansion $M(x,v,\tau,n) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} C_{ij}^{\;n}(\tau) x^i v^j$

  terminal condition $M(x,v,T) = x^n$

- With the SDE of volatility, formulate the backward equation

$$\tfrac{1}{2}vM_{xx} + \rho\xi v M_{xv} + \tfrac{1}{2}\xi^2 v M_{vv} + (\mu - \tfrac{1}{2}v)M_x + k(\theta - v)M_v = M_\tau$$

- Solve the coefficients C

Approach

- Fourier Transform based solution
  - Heston(1993) gave a solution based on two parts

$$C(s,v,t) = SP_1 - KP(t,T)P_2$$

1st part: present value of the spot asset before optimal exercise

2nd part: present value of the strike-price payment

  - P1 and P2 satisfy the backward equation, thus their characteristic function also satisfy the backward equation

- Measure the distance of these two solutions to determine the highest moment we need to use

Implementation

- Software: Mathematical 5.1
  Matlab 7.5

- Platform: Mobile (labtop)
  GRACE(consists of five Sun servers)
  www.grace.umd.edu

Validation/Testing

- Give $v_t = \theta$ and $\xi = 0$, Model should have Black-Scholes Model solutions.
- Estimate Five Parameters: $v_0, \xi, \theta, \kappa, \rho$
- Minimize the Sum of Square Error function

$$SSE(\kappa, v_0, \xi, \rho, \theta) = \sum_{i=1}^{n} \{\sigma_i^{'} - \sigma_i(\kappa, v_0, \xi, \rho, \theta)\}^2$$

$\sigma_i^{'}$ is the observation

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